SOME DICHOTOMY THEOREMS

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1. The G_0 dichotomy

A *digraph* (or directed graph) on a set X is a subset $G \subseteq X^2 \setminus \Delta$. Given a digraph G on a set X and a subset $A \subseteq X$, we say that A is *G-discrete* if for all $x, y \in A$ we have $(x, y) \notin G$.

Now let $s_n \in 2^n$ be chosen for every $n \in \mathbb{N}$ such that $\forall s \in 2^{<\mathbb{N}}\ \exists n\ s \sqsubseteq s_n$. Then we can define a digraph G_0 on $2^{\mathbb{N}}$ by

$$G_0 = \{(s_n 0x, s_n 1x) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} \mid n \in \mathbb{N}\ \&\ x \in 2^{\mathbb{N}}\}.$$

Exercise 1. Show that if $x, y \in 2^{\mathbb{N}}$ differ in only finitely many coordinates, then there is a path $x_0 = x, x_1, \ldots, x_n = y$ such that for all i, either $(x_i, x_{i+1}) \in G_0$ or $(x_{i+1}, x_i) \in G_0$. *Hint:* The proof is by induction on the last coordinate in which they differ.

Lemma 2. *If $f : 2^{\mathbb{N}} \to X$ is a continuous function into a Polish space X such that $xG_0y \Rightarrow f(x) = f(y)$, then f is constant.*

Proof. If not, by continuity, we can find basic open sets $N_s, N_t \subseteq 2^{\mathbb{N}}$ such that $f[N_s] \cap f[N_t] = \emptyset$. Extending s or t, we can suppose that $|s| = |t|$, and thus for any $x \in 2^{\mathbb{N}}$, $f(sx) \neq f(tx)$. On the other hand, such sx and tx differ only in finitely many coordinates, so by Exercise 1 they are connected by a path in G_0, which contradicts the properties of f. □

Lemma 3. *If $B \subseteq 2^{\mathbb{N}}$ has the Baire property and is non-meagre, then B is not G_0-discrete.*

Proof. By assumption on B, we can find some $s \in 2^{<\mathbb{N}}$ such that B is comeagre in N_s. Also, by choice of (s_n), we can find some n such that $s \sqsubseteq s_n$, whereby B is comeagre in N_{s_n}. By the characterisation of comeagre subsets of $2^{\mathbb{N}}$, we see that for some $x \in 2^{\mathbb{N}}$, we have $s_n 0x, s_n 1x \in B$, showing that B is not G_0-discrete. □

Suppose G and H are digraphs on sets X and Y respectively. A *homomorphism* from G to H is a function $h : X \to Y$ such that for all $x, y \in X$,

$$(x, y) \in G \Rightarrow (h(x), h(y)) \in H.$$

Also, if Z is any set, a *Z-colouring* of a digraph G on X is a homomorphism from G to the digraph $\neq$ on Z, i.e., a function $h : X \to Z$ such that for all $x, y \in X$,

$$(x, y) \in G \Rightarrow h(x) \neq h(y).$$

Proposition 4. *There is no Baire measurable $\mathbb{N}$-colouring of G_0.*

Proof. Note that if $h : 2^{\mathbb{N}} \to \mathbb{N}$ is a Baire measurable function, then for some $n \in \mathbb{N}$, $B = h^{-1}(n)$ is non-meagre with the Baire property and hence not G_0-discrete. So h cannot be a homomorphism from G_0 to $\neq$ on $\mathbb{N}$. □

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Theorem 5 (Kechris–Solecki–Todorcevic)**.** *Suppose G is an analytic digraph on a Polish space X. Then exactly one of the following holds:*

- *there is a continuous homomorphism from G_0 to G,*
- *there is a Borel $\mathbb{N}$-colouring of G.*

Proof. (B. Miller) If X is countable, the result is trivial. So if not, let $f: \mathbb{N}^{\mathbb{N}} \to P$ be a continuous bijection onto the perfect kernel P of X. By replacing G with $(f \times f)^{-1}[G]$, there is no loss of generality in assuming that $X = \mathbb{N}^{\mathbb{N}}$.

So suppose $F \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is a closed set such that

$$(x,y) \in G \Leftrightarrow \exists z\,(x,y,z) \in F.$$

In order to produce a continuous homomorphism h from G_0 to G it suffices to find monotone Lipschitz functions $u, v^m : 2^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$, $m \in \mathbb{N}$, such that for all $m < k$ and $t \in 2^{k-m-1}$,

$$\big(N_{u(s_m 0t)} \times N_{u(s_m 1t)} \times N_{v^m(t)}\big) \cap F \neq \emptyset.$$

In this case, we can define $h, \tilde{v}^m : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by $h(w) = \bigcup_n u(w|_n)$ and $\tilde{v}^m(w) = \bigcup_n v^m(w|_n)$. For then if $m \in \mathbb{N}$ and $w \in 2^{\mathbb{N}}$ are given, there are $x_k, y_k, z_k \in \mathbb{N}^{\mathbb{N}}$ such that $x_k \to h(s_m 0w)$, $y_k \to h(s_m 1w)$ and $z_k \to \tilde{v}^m(w)$ such that for all k, (x_k, y_k, z_k). So, as F is closed, also

$$(h(s_m 0w), h(s_m 1w), \tilde{v}^m(w)) \in F,$$

whence $(h(s_m 0w), h(s_m 1w)) \in G$, showing that h is a homomorphism from G_0 to G.

An *n-approximation* is a pair (u,v) of functions $u: 2^n \to \mathbb{N}^n$ and $v: 2^{<n} \to \mathbb{N}^n$. Also, if (u,v) is an n-approximation and (u',v') is an $n+1$-approximation, we say that (u',v') *extends* (u,v) if $u(s) \sqsubseteq u'(si)$ and $v(t) \sqsubseteq v'(ti)$ for all $s \in 2^n$, $t \in 2^{<n}$ and $i = 0,1$.

Suppose $A \subseteq X$ and (u,v) is an n-approximation. We define the set of A-realisations, $\mathbb{R}(A,u,v)$, to be the set of pairs of tuples $(x_s)_{s\in 2^n} \in \prod_{s\in 2^n} \big(A \cap N_{u(s)}\big)$ and $(z_t)_{t \in 2^{<n}} \in \prod_{t\in 2^{<n}} N_{v(t)}$ such that

$$(x_{s_m 0t}, x_{s_m 1t}, z_t) \in F$$

for all $s \in 2^n$, $m \in \mathbb{N}$ and $t \in 2^{n-m-1}$. So if (u_0, v_0) is the unique 0-approximation (i.e., $u(\emptyset) = \emptyset$ and v is the function with empty domain), we have $\mathbb{R}(A, u_0, v_0) = \{x_\emptyset \mid x_\emptyset \in A\} = A$. If (u,v) has no A-realised extension, we say that (u,v) is *A-terminal*.

Lemma 6. *Suppose (u,v) is an A-terminal n-approximation, then*

$$\mathbb{D}(A,u,v) = \{x_{s_n} \mid ((x_s)_{s\in 2^n}, (z_t)_{t\in 2^{<n}}) \in \mathbb{R}(A,u,v)\}$$

is G-discrete.

Proof. Suppose toward a contradiction that

$$((x^0_s)_{s\in 2^n}, (z^0_t)_{t\in 2^{<n}}),\ ((x^1_s)_{s\in 2^n}, (z^1_t)_{t\in 2^{<n}}) \in \mathbb{R}(A,u,v)$$

satisfy $(x^0_{s_n}, x^1_{s_n}) \in G$. Then for some $z_\emptyset \in \mathbb{N}^{\mathbb{N}}$, we have

$$(x^0_{s_n}, x^1_{s_n}, z_\emptyset) \in F,$$

and hence, setting $x_{si} = x^i_s$ and $z_{ti} = z^i_t$ for all $si \in 2^{n+1}$ and $ti \in 2^{<n+1} \setminus \{\emptyset\}$, we get an A-realisation $((x_s)_{s\in 2^{n+1}}, (z_t)_{t\in 2^{<n+1}})$ of an extension of (u,v), contradicting that (u,v) is A-terminal. $\square$

Now define $\Phi \subseteq P(X)$ by

$$\Phi(A) \Leftrightarrow A \text{ is } G\text{-discrete}.$$

Since G is analytic, Φ is Π^1_1 on Σ^1_1, and so, by the First Reflection Theorem, any G-discrete analytic set A is contained in a G-discrete Borel set A'. Using this, we can define a function D assigning to each Borel set $A \subseteq X$ a Borel subset given by

$$D(A) = A \setminus \bigcup \{\mathbb{D}(A,u,v)' \mid (u,v) \text{ is } A\text{-terminal }\}.$$

Note that, as there are only countably many approximations (u,v), the set $A \setminus D(A)$ is a countable union of G-discrete Borel sets.

Lemma 7. *Suppose (u,v) is an n-approximation all of whose extensions are A-terminal. Then (u,v) is $D(A)$-terminal.*

Proof. Note that if (u,v) is not $D(A)$-terminal, there is some extension (u',v') of (u,v) and some realisation $((x_s)_{s\in 2^{n+1}},(z_t)_{t\in 2^{<n+1}}) \in \mathbb{R}(D(A),u',v') \subseteq \mathbb{R}(A,u',v')$. But since (u',v') is A-terminal, we have $\mathbb{D}(A,u',v') \cap D(A) = \emptyset$, contradicting that $\phi(x_{s_{n+1}}) \in \mathbb{D}(A,u',v') \cap D(A)$. □

Now define, by transfinite induction, $D^0(X) = X$, $D^{\xi+1}(X) = D(D^\xi(X))$ and $D^\lambda(X) = \bigcap_{\xi<\lambda} D^\xi(X)$, whenever λ is a limit ordinal. Then $(D^\xi(X))_{\xi<\omega_1}$ is a well-ordered, decreasing sequence of Borel subsets of X, so the sets T_ξ of approximations (u,v) that are $D^\xi(X)$-terminal is an increasing sequence of subsets of the countable set of all approximations. It follows that for some $\xi < \omega_1$, we have $T_\xi = T_{\xi+1}$.

Now if $(u,v) \notin T_{\xi+1}$, then (u,v) is not $D(D^\xi(X))$-terminal and hence admits an extension (u',v') that is not $D^\xi(X)$-terminal either, whereby $(u',v') \notin T_\xi = T_{\xi+1}$. So if (u_0,v_0) denotes the unique 0-approximation and $(u_0,v_0) \notin T_{\xi+1}$, we can inductively construct $(u_n,v_n) \notin T_{\xi+1}$ extending each other. Setting

$$u = \bigcup_n u_n$$

and for $t \in 2^n$

$$v^m(t) = v_{n+m+1}(t),$$

we have the required monotone Lipschitz functions $u, v^m : 2^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ to produce a continuous homomorphism from G_0 to G.

Conversely, if $(u_0,v_0) \in T_{\xi+1}$, then (u_0,v_0) is $D^{\xi+1}(X)$-terminal and hence $D^{\xi+2}(X) \subseteq D^{\xi+1}(X) \setminus \mathbb{D}(D^{\xi+1}(X),u_0,v_0)$. But, since (u_0,v_0) is the unique 0-approximation, we have

$$\mathbb{D}(D^{\xi+1}(X),u_0,v_0) = \mathbb{R}(D^{\xi+1}(X),u_0,v_0) = D^{\xi+1}(X),$$

whereby $D^{\xi+2}(X) = \emptyset$. It follows that

$$X = \bigcup_{\zeta<\xi+2} D^\zeta(X) \setminus D^{\zeta+1}(X)$$

is a countable union of G-discrete Borel sets. We can then define a Borel $\mathbb{N}$-colouring of G by letting $c(x)$ be a code for the discrete Borel subset of X to which x belongs. □

Exercise 8. By inspection of the proof of Theorem 5, show that if G is a κ-Souslin digraph on $\mathbb{N}^\mathbb{N}$, then one of the following holds

- there is a continuous homomorphism from G_0 to G,
- there is a κ-colouring of G.

2. The Mycielski, Silver and Burgess dichotomies

Theorem 9 (Mycielski's Independence Theorem)**.** *Suppose X is a perfect Polish space and $R \subseteq X^2$ is a comeagre set. Then there is a continuous injection $\phi\colon 2^{\mathbb{N}} \to X$ such that for all distinct $x, y \in 2^{\mathbb{N}}$ we have $(\phi(x), \phi(y)) \in R$.*

Proof. Let $d \leq 1$ be a compatible complete metric on X and choose a decreasing sequence of dense open subsets $U_n \subseteq X^2$ such that $\bigcap_{n\in\mathbb{N}} U_n \subseteq R$. We construct a Cantor scheme $(C_s)_{s\in 2^{<\mathbb{N}}}$ of non-empty open subsets of X by induction on the length of s such that for all distinct $s, t \in 2^n$ and $i = 0, 1$, we have

$$\overline{C}_{si} \subseteq C_s, \quad \mathrm{diam}(C_s) \leq \frac{1}{|s|+1}, \quad \text{and} \quad C_s \times C_t \subseteq U_{n-1}.$$

To see how this is done, suppose that C_s has been defined for all $s \in 2^n$. Since X is perfect, we can find disjoint, non-empty open subsets D_{s0} and D_{s1} of C_s for every $s \in 2^n$. Now, as U_n is dense, $U_n \cap (D_t \times D_{t'}) \neq \emptyset$ for all distinct $t, t' \in 2^{n+1}$ and so we can inductively shrink the D_t to open subsets C_t such that whenever $t, t' \in 2^{n+1}$ are distinct, we have $C_t \times C_{t'} \subseteq U_n$. By further shrinking the C_{si} if necessary, we can ensure that $\overline{C}_{si} \subseteq C_s$ and $\mathrm{diam}(C_s) \leq \frac{1}{|s|+1}$. Now letting $\phi\colon 2^{\mathbb{N}} \to X$ be defined by $\{\phi(x)\} = \bigcap_{n\in\mathbb{N}} C_{x|_n}$, we see that ϕ is continuous. Also, if $x, y \in 2^{\mathbb{N}}$ are distinct, then for all but finitely many n we have $(\phi(x), \phi(y)) \in C_{x|_n} \times C_{y|_n} \subseteq U_{n-1}$, so, since the U_n are decreasing, we have $(x, y) \in \bigcap_{n\in\mathbb{N}} U_n \subseteq R$. □

Theorem 10 (J. Silver)**.** *Suppose E is a conalytic equivalence relation on a Polish space X. Then exactly one of the following holds*

- *E has at most countably many classes,*
- *there is a continuous injection $\phi\colon 2^{\mathbb{N}} \to X$ such that for distinct $x, y \in 2^{\mathbb{N}}$, $\neg\phi(x) E \phi(y)$.*

Proof. We define an analytic digraph G on X by setting $G = X^2 \setminus E$. Notice first that if $c\colon X \to \mathbb{N}$ is a Borel $\mathbb{N}$-colouring of G, then for all $x, y \in X$,

$$\neg x E y \Rightarrow (x, y) \in G \Rightarrow c(x) \neq c(y).$$

So for any $n \in \mathbb{N}$, $c^{-1}(n)$ is contained in a single equivalence class of E. Moreover, as $X = \bigcup_{n\in\mathbb{N}} c^{-1}(n)$, this shows that X is covered by countably many E-equivalence classes.

So suppose instead that there is no Borel $\mathbb{N}$-colouring of G. Then by Theorem 5 there is a continuous homomorphism $h\colon 2^{\mathbb{N}} \to X$ from G_0 to G. Now let $F = \{(x, y) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} \mid h(x) E h(y)\}$. Then F is meagre. For otherwise, by the Kuratowski–Ulam Theorem, there is some $x \in 2^{\mathbb{N}}$ such that F_x is non-meagre and hence, by Lemma 3, there are $y, z \in F_x$ such that $(y, z) \in G_0$. As h is a homomorphism it follows that $(h(y), h(z)) \in G = X^2 \setminus E$, which contradicts that $h(y) E h(x) E h(z)$. Therefore, applying Mycielski's Theorem to the meagre set F, we get a continuous function $f\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for distinct $x, y \in 2^{\mathbb{N}}$, $(f(x), f(y)) \notin F$, i.e., $\neg h \circ f(x) E h \circ f(y)$. Letting $\phi = h \circ f$, we have the result. □

By the same proof, using istead the G_0-dichotomy for ω_1-Souslin sets, we deduce the following result.

Theorem 11 (J. Burgess, L. A. Harrington–S. Shelah)**.** *Let E be a Σ^1_2 equivalence relation on a Polish space X. Then one of the following holds*

- *E has at most $\aleph_1$ classes,*

- *there is a continuous injection* $\phi\colon 2^{\mathbb{N}} \to X$ *such that for distinct* $x, y \in 2^{\mathbb{N}}$, $\neg\phi(x)E\phi(y)$.

Now as the isomorphism relation between the countable models of an $L_{\omega_1\omega}$-sentence is an analytic equivalence relation, we have the following corollary, initially proved by analysing the space of complete types.

Corollary 12 (M. Morley)**.** *Suppose* L *is a countable language and* σ *is a* $L_{\omega_1\omega}$ *sentence. Then there are either a continuum of non-isomorphic countable models of* σ *or at most* $\aleph_1$ *non-isomorphic models of* σ.

Theorem 13 (Lusin–Novikov)**.** *Suppose* X *and* Y *are Polish spaces and* $A \subseteq X \times Y$ *a Borel subset. Assume that for every* $x \in X$*, the vertical section* A_x *is countable. Then there are Borel sets* F_n *such that* $|(F_n)_x| \leqslant 1$ *for every* $x \in X$ *and* $A = \bigcup_{n\in\mathbb{N}} F_n$.

Proof. Define a Borel digraph G on $X \times Y$ by

$$(x, y)G(x', y') \Leftrightarrow x = x' \;\&\; y \neq y' \;\&\; (x, y) \in A \;\&\; (x', y') \in A.$$

Assume first that $f\colon X \times Y \to \mathbb{N}$ is a Borel $\mathbb{N}$-coloring of G. Then for every $n \in \mathbb{N}$, $F_n = A \cap f^{-1}(n)$ is a G-discrete Borel subset of $X \times Y$ and $A = \bigcup_{n\in\mathbb{N}} F_n$. Moreover, since F_n is G-discrete, we see that $|(F_n)_x| \leqslant 1$ for all $x \in X$.

Now, by Theorem 5, if there is no such colouring, then there is a continuous homomorphism $h\colon 2^{\mathbb{N}} \to X \times Y$ from G_0 to G. Composing with the coordinate projections, we obtain continuous functions $h_X\colon 2^{\mathbb{N}} \to X$ and $h_Y\colon 2^{\mathbb{N}} \to Y$ such that

$$aG_0b \Rightarrow h_X(a) = h_X(b).$$

By Lemma 2, h_X is constant with some value $x_0 \in X$ and so

$$aG_0b \Rightarrow h_Y(a) \neq h_Y(b) \;\&\; h_Y(a) \in A_{x_0} \;\&\; h_Y(b) \in A_{x_0}.$$

Since A_{x_0} is countable, there is an injection $\pi\colon A_{x_0} \to \mathbb{N}$, and thus $\pi \circ h_Y\colon 2^{\mathbb{N}} \to \mathbb{N}$ is a continuous $\mathbb{N}$-colouring of G_0, contradicting Proposition 4. So the first option holds. $\square$